TYPE - III :→ Equation of the form

$$f_1(x, p) = f_2(y, q) \quad \text{i.e seperable eqn}$$

As a trial solution
put each side is equal to 'a'
arbitrary constant.

i.e $f_1(x, p) = f_2(y, q) = a$

In the above equation finding
p & q, put in

$$dz = p\,dx + q\,dy$$

integrating & we get the
complete solⁿ.

(Q1) $p^2 + q^2 = x + y$

Solⁿ :→ It is of the form $f_1(x, p) = f_2(y, q)$

Let $p^2 - x = y - q^2 = c$

obviously, $p = \sqrt{c+x}$, $q = \sqrt{y-c}$

Putting $dz = p\,dx + q\,dy$, we get

$$\therefore dz = \sqrt{c+x}\,dx + \sqrt{y-c}\,dy$$

integrating

$$z = \frac{2}{3}(c+x)^{3/2} + \frac{2}{3}(y-c)^{3/2} + c_1$$ is

the complete solⁿ.

(Q2) Solve $p.q = xy$

Solⁿ:- $\frac{p}{x} = \frac{y}{q} = c$

$\therefore p = cx, \quad q = \frac{y}{c}$

putting $dz = p\,dx + q\,dy$, we get

$\therefore dz = cx\,dx + \frac{y}{c}\,dy$

integrating

$$z = \frac{c}{2}x^2 + \frac{1}{2c}y^2 + c_1$$ is the complete solⁿ.

imp (Q3) $z^2(p^2+q^2) = x^2+y^2$

Solⁿ:→ Putting $z\,dz = dZ$

$\therefore \frac{z^2}{2} = Z$

The given equation reduced to

$P^2 + Q^2 = x^2 + y^2$, where $P = \frac{dZ}{dx}$

$Q = \frac{dZ}{\partial y}$

$\therefore p^2 - x^2 = -q^2 + y^2 = a$ (say)

$\therefore P = \sqrt{a+x^2}$ & $q = \sqrt{y^2 \pm a}$

Putting $dz = P\,dx + q\,dy$

$\therefore dz = \sqrt{a+x^2}\,dx + \sqrt{y^2-a}\,dy$

integrating we get

$$z = \frac{x}{2}\sqrt{a+x^2} + \frac{a}{2}\log\{x+\sqrt{a+x^2}\}$$

$$+ \frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\log\{y+\sqrt{y^2-a^2}\} + b$$

$$\therefore z^2 = x\sqrt{a+x^2} + a\log\{x+\sqrt{a+x^2}\}$$

$$+ y\sqrt{y^2-a} - a\log\{y+\sqrt{y^2-a})\} + c$$

(8) $pe^y = qe^x$

Sol^n:→ The given equation can be written as

$pe^{-x} = qe^{-y} = a$ (say)

$\therefore p = ae^x$ & $q = ae^y$

Putting these values in

$dz = p\,dx + q\,dy$

$\therefore dz = ae^x dx + ae^y dy$

integrating, we get

$z = ae^x + ae^y + c$

~~(9)~~ $z(p^2 - q^2) = x - y$

Sol^n:→ The given equation can be written as

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = x - y$$

Putting $\sqrt{z}\,dz = dZ \quad \therefore Z = \frac{2}{3} z^{\frac{3}{2}}$

$\therefore$ The equation becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

or, $P^2 - Q^2 = x - y$, where $P = \frac{\partial Z}{\partial x}$

$Q = \frac{\partial Z}{\partial y}$

$\therefore P^2 - x = Q^2 - y = a$

$\therefore P = \sqrt{x+a}$ & $Q = \sqrt{y+a}$

Putting $dZ = P\,dx + Q\,dy$

$\therefore dZ = \sqrt{x+a}\,dx + \sqrt{y+b}\,dy$

integrating

$$Z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{\frac{3}{2}} + b$$

(Q) Find the compete integral of

$q = px + p^2 = a$

Sol^n: $\rightarrow$ $q = a$

$p^2 + xp - a = 0$

$$p = \frac{-x \pm (x^2+4a)^{3/2}}{2}$$

$\therefore dz = p\,dx + q\,dy$

$$\therefore z = -\frac{x^2}{4} + \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a \log\{x + \sqrt{x^2+4a}\}\right.$$

$+ a^4 + b$